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RIESZ TYPE MINIMAL OMEGA-REPRESENTATIONS IN THE HALF-PLANE

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Abstract

First, some Green type potentials are introduced in the upper half-plane, which depend on a functional parameter $\omega(x)$ given on $(0, +\infty)$ and can have any mass density near the finite points of the real axis. These potentials possess a minimality property in the sense that they coincide with the ordinary Green potentials in the upper half-plane after application of some generalization of Liouville's fractional integration. Then, the Riesz type descriptive representations of some Nevanlinna-Djrbashian type classes of functions delta-subharmonic in the half-plane and possessing there bounded Tsuji characteristics are established, where the new potentials participate and an analogue of the Stieltjes inversion formula is true.

1. Introduction

This paper is devoted to the descriptive Riesz type representations of some classes of functions delta-subharmonic in the upper half-plane $G^+ = \{z : \operatorname{Im} z > 0\}$ and possessing there bounded Tsuji characteristics.

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These classes and their representations, in a sense, are similar to those investigated by Djrbashian in the unit disc of the complex plane [1] (see also [2, 5]). Besides, the representations of this paper differ from those of [8], though as those of [8], they can have any mass density near the finite points of the real axis. The main difference is that the representations of this paper posses a minimality property, which is revealed by application of the operator

$$L_\omega f(z) = \int_0^{+\infty} f(z+it)d\omega(t), \quad z \in G^+, \tag{1.1}$$

becoming the Liouville fractional integration when $\omega(t) = t^\alpha/\Gamma(1+\alpha)$ $(\alpha > 0)$. It is easy to see that the Djrbashian kernel

$$C_\omega(z) = \int_0^{+\infty} e^{itz} \frac{dt}{I_\omega(t)}, \quad I_\omega(t) = \int_0^{+\infty} e^{-tx} d\omega(x), \tag{1.2}$$

is transferred by L_ω to the ordinary Cauchy kernel, i.e.,

$$L_\omega C_\omega(z) = \frac{1}{-iz} \equiv C_0(z), \quad z \in G^+, \tag{1.3}$$

for "good enough" functions $\omega(x)$. Besides, $C_\omega(z) = (-iz)^{1+\alpha}$ for $\omega(t) = t^\alpha$ $(\alpha > 0)$. Note that, being an obvious generalization of the ordinary Cauchy kernel in the one-dimensional case, the ω-kernel (1.2) was first used in [9], where it was constructed in the multidimensional case of tube domains.

Everywhere below, we assume that $\omega(x)$ is a continuously differentiable function in some interval $[0, \Delta] \subset (0, +\infty)$, such that $\omega(0) = 0$, $\omega(x) = \omega(\Delta)\,(\Delta < x < +\infty)$, $\omega'(x) > 0\,(0 < x < \Delta)$, and

$$\int_0^\Delta \omega(x) \frac{dx}{x} < +\infty.$$

In first four sections, if this paper, some Green type potentials in the upper half-plane, depending on the functional parameter $\omega(x)\,(0 < x < +\infty)$ are introduced and investigated. In difference to those of [8], these potentials possess a minimality property in the sense that, they coincide with the ordinary Green potentials in the upper half-plane after application of the operator L_ω. Then, in two subsequent sections, the Riesz type descriptive representations of some Nevanlinna-Djrbashian type classes of functions delta-subharmonic in the half-plane and possessing there bounded Tsuji characteristics are found, where the new potentials participate and an analogue of the Stieltjes inversion formula is true.

2. Blaschke Type Factors

2.1. Assuming that $\zeta = \xi + i\eta \in G^+$ is a fixed point, for $\operatorname{Im} z > \eta$ introduce the Blaschke type factors

$$\widetilde{b}_\omega(z, \zeta) = \exp\left\{-\int_0^\eta [C_\omega(z - \zeta + it) + C_\omega(z - \overline{\zeta} + it)]\omega(t)dt\right\}. \tag{2.1}$$

It is easy to see that

$$\widetilde{b}_\omega(z, \zeta) = b_\omega(z, \zeta)\exp\{-V_\omega(z, \zeta)\},$$

where $b_\omega(z, \zeta)$ is the Blaschke factor introduced in [8] and

$$V_\omega(z, \zeta) = -\int_\eta^{2\eta} C_\omega(z - \zeta + it)\omega(t)dt + \int_0^\eta C_\omega(z - \overline{\zeta} - it)\omega(t)dt, \tag{2.2}$$

is a holomorphic function in G^+. Besides, it is not difficult to verify that the following formula is true for the ordinary Blaschke factor:

$$\log|b_0(z, \zeta)| \equiv \log\left|\frac{z - \zeta}{z - \overline{\zeta}}\right| = -\operatorname{Re}\int_0^\eta \left[\frac{1}{t - i(z - \zeta)} - \frac{1}{t + i(z - \overline{\zeta})}\right]dt. \tag{2.3}$$

First, we prove the following lemma:

Lemma 2.1. *If* $\zeta = \xi + i\eta \in G^+$ *is a fixed point, then*

$$\widetilde{b}_\omega(z, \zeta) = b_0(z, \zeta)\exp\{-J_\omega(z, \zeta)\}, \quad z \in G^+, \tag{2.4}$$

where

$$J_\omega(z,\ \zeta) = \frac{1}{\pi}\int_{-\infty}^{+\infty} C_\omega(z-u) L_\omega \log|b_0(u,\ \zeta)| du, \tag{2.5}$$

is a function holomorphic in G^+.

Proof. Using formula (2.3), one can verify that for any $u \in (-\infty,\ +\infty)$ and $t > 0$,

$$|\log|b_0(u+it,\ \zeta)|| = \operatorname{Re}\int_{-\eta}^{\eta}\frac{d\sigma}{\sigma+t+i(u-\xi)} = \int_{-\eta}^{\eta}\frac{(\sigma+t)d\sigma}{(\sigma+t)^2+(u-\xi)^2}. \tag{2.6}$$

Hence, for any $t > 0$,

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}|\log|b_0(u+it,\ \zeta)||du$$

$$= \int_{-\eta}^{\eta} d\sigma\,\frac{\sigma+t}{\pi}\int_{-\infty}^{+\infty}\frac{du}{(\sigma+t)^2+(u-\xi)^2}$$

$$= \begin{cases} \displaystyle\int_{-\eta+t}^{\eta+t} d\lambda\,\frac{\lambda}{\pi}\int_{-\infty}^{+\infty}\frac{du}{\lambda^2+(u-\xi)^2}, & \text{when } \eta < t < +\infty, \\ \displaystyle\left(\int_0^{\eta+t} - \int_{-\eta+t}^{-0}\right) d\lambda\,\frac{\lambda}{\pi}\int_{-\infty}^{+\infty}\frac{du}{\lambda^2+(u-\xi)^2}, & \text{when } 0 < t < \eta, \end{cases}$$

and

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}|\log|\, b_0(u+it,\ \zeta)||du = \begin{cases} 2\eta, & \text{when } \eta < t < +\infty, \\ 2t, & \text{when } 0 < t < \eta. \end{cases} \tag{2.7}$$

Further, using formulas (2.5), (2.7), and the estimate (3.15) of [7], we conclude that for any $z = x + iy \in G^+$ and $\delta \in (0,\ 1)$

$$|J_\omega(z,\ \zeta)| \le \frac{1}{\pi}\int_{-\infty}^{+\infty}|C_\omega(z-u)|du\int_0^{+\infty}|\log|b_0(u+it,\ \zeta)||d\omega(t)$$

$$\le \frac{M_{y,\,\delta}}{\pi}\int_{-\infty}^{+\infty}\frac{du}{|z-u|^{1-\delta}}\int_0^{+\infty}|\log|b_0(u+it,\ \zeta)||d\omega(t)$$

$$\leq 2\frac{M_{y,\,\delta}}{y^{1-\delta}}\left[\eta\int_{\eta}^{+\infty}d\omega(t)+\int_{0}^{\eta}td\omega(t)\right]\leq M_{y,\,\delta}^{*}\omega(\Delta)\eta, \tag{2.8}$$

where $M_{y,\delta}^{*} > 0$ is a constant depending only on y and δ, and it is obvious that the integrand in $J_\omega(z, \zeta)$ has an independent of $z \in G^+$ integrable majorant, provided $y > 0$ does not approach zero. Hence, the function $J_\omega(z, \zeta)$ is holomorphic in G^+. Besides, by the Fubini theorem,

$$J_\omega(z,\,\zeta)=\int_0^\Delta d\omega(t)\frac{1}{\pi}\int_{-\infty}^{+\infty}C_\omega(z-u)\log|b_0(u+it,\,\zeta)|\,du$$

$$\equiv\int_0^\Delta A_\omega(z,\,\zeta,\,t)d\omega(t),$$

where the inner integral is absolutely convergent. Therefore, by (2.6),

$$A_\omega(z,\,\zeta,\,t)=-\int_{-\eta}^{\eta}d\sigma\frac{1}{\pi}\int_{-\infty}^{+\infty}C_\omega(z-u)\,\mathrm{Re}\left\{\frac{1}{\sigma+t+i(u-\xi)}\right\}du,$$

provided $\sigma \neq t$. Now, calculating the integral

$$K_\omega(z,\,\zeta,\,t)\equiv\frac{1}{\pi}\int_{-\infty}^{+\infty}C_\omega(z-u)\,\mathrm{Re}\left\{\frac{1}{\sigma+t+i(u-\xi)}\right\}du$$

$$=\frac{1}{\pi}\int_{-\infty}^{+\infty}C_\omega(z+u)\,\mathrm{Re}\left\{\frac{1}{\sigma+t-i(u+\xi)}\right\}du$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}C_\omega(z+u)\left\{\frac{1}{\sigma+t+i(u+\xi)}+\frac{1}{\sigma+t-i(u+\xi)}\right\}du,$$

we obviously get

$$K_\omega(z,\,\zeta,\,t)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{C_\omega(z+u)du}{u-[-\xi+i(\sigma+t)]}-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{C_\omega(z+u)du}{u-[-\xi-i(\sigma+t)]}.$$

Besides, it follows from the estimate (3.2) of [7] that for any fixed $z \in G^+$, the function $C_\omega(z+w)$ belongs to the Hardy H^2 in G^+. Hence by Cauchy's formula,

$$K_\omega(z,\,\zeta,\,t)=\begin{cases}C_\omega(z-\xi+i(\sigma+t)), & \text{when }\ \sigma+t>0,\\ -\,C_\omega(z-\xi-i(\sigma+t)), & \text{when }\ \sigma+t<0.\end{cases}$$

Thus,

$$-J_\omega(z,\,\zeta)=\int_{-\eta}^{\eta}d\sigma\int_{(-\sigma)^+}^{+\infty}C_\omega(z-\xi+i(\sigma+t))d\omega(t)$$

$$-\int_{-\eta}^{\eta}d\sigma\int_{0}^{(-\sigma)^+}C_\omega(z-\xi-i(\sigma+t))d\omega(t)$$

$$=\int_0^{\eta}d\sigma\int_0^{+\infty}C_\omega(z-\xi+i(\sigma+t))d\omega(t)$$

$$+\int_{-\eta}^{0}d\sigma\int_0^{+\infty}C_\omega(z-\xi+i(\sigma+t))d\omega(t)$$

$$-\int_{-\eta}^{0}d\sigma\int_0^{-\sigma}C_\omega(z-\xi-i(\sigma+t))d\omega(t)$$

$$-\int_{-\eta}^{0}d\sigma\int_0^{-\sigma}C_\omega(z-\xi+i(\sigma+t))d\omega(t).$$

Therefore, using formula (1.3) and integrating by parts, we get

$$-J_\omega(z,\,\zeta)=\int_0^{\eta}[C_0(z-\overline{\zeta}-i\tau)+C_0(z-\zeta+i\tau)]d\tau$$

$$-\int_0^{\eta}\left[\int_t^{\eta}C_\omega(z-\xi+it-i\sigma)d\sigma\right]d\omega(t)$$

$$-\int_0^{\eta}\left[\int_t^{\eta}C_\omega(z-\xi-it+i\sigma)d\sigma\right]d\omega(t)$$

$$=\int_0^{\eta}[C_0(z-\overline{\zeta}-i\lambda)+C_0(z-\zeta+i\lambda)]d\lambda$$

$$-\int_0^{\eta}\left[\int_0^{\eta-t}C_\omega(z-\xi-i\lambda)d\lambda\right]d\omega(t)$$

$$-\int_0^{\eta}\left[\int_0^{\eta-t} C_\omega(z-\xi+i\lambda)d\lambda\right]d\omega(t)$$

$$=\int_0^{\eta}\frac{d\tau}{\tau-i(z-\zeta)}-\int_0^{\eta}\frac{d\tau}{\tau+i(z-\overline{\zeta})}$$

$$-\int_0^{\eta} C_\omega(z-\zeta+it)\omega(t)dt-\int_0^{\eta} C_\omega(z-\overline{\zeta}-it)\omega(t)dt$$

$$=-\log b_0(z,\,\zeta)+\log \widetilde{b}_\omega(z,\,\zeta).$$

Hence, the desired representation (2.4) holds.

Also, the following theorem on the properties of the factors $\widetilde{b}_\omega(z,\,\zeta)$ is true:

Theorem 2.1. *For any fixed* $\zeta=\xi+i\eta\in G^+$, *the function* $\widetilde{b}_\omega(z,\,\zeta)$ *is holomorphic in* G^+, *where it has a unique, first order zero at the point* $z=\zeta$.

Proof. If $\zeta=\xi+i\eta\in G^+$ is fixed and $\operatorname{Im} z\geq\rho$ for some $\rho>0$, then by (1.2),

$$|C_\omega(z-\zeta+it)|\leq C_\omega(i(y-\eta+t))\leq C_\omega(i\rho)<+\infty,\quad\text{when}\quad \eta\leq t\leq 2\eta,$$

$$|C_\omega(z-\overline{\zeta}-it)|\leq C_\omega(i(y+\eta-t))\leq C_\omega(i\rho)<+\infty,\quad\text{when}\quad 0\leq t\leq \eta.$$

Thus, the integrands in (2.1), which are holomorphic in G^+, have independent of $z(\operatorname{Im} z\geq\rho)$, integrable majorants. Consequently, the function $V_\omega(z,\,\zeta)$ is holomorphic in any half-plane $\operatorname{Im} z\geq\rho$, and hence in the whole G^+.

2.2. Below, we study some properties of the Blaschke type factor $\widetilde{b}_\omega(z,\,\zeta)$, which are revealed by application of the operator L_ω of (1.1). To this end, we shall often use the representations of $\log|\widetilde{b}_\omega(z,\,\zeta)|$ given in the next two lemmas.

Lemma 2.2. *If* $\zeta = \xi + i\eta \in G^+$ *is a fixed point, then the function* $L_\omega \log|\widetilde{b}_\omega(z,\ \zeta)|$ *is harmonic everywhere in the finite complex plane, except the straight line closed interval* $[\zeta,\ \overline{\zeta}]$ *with endpoints* ζ *and* $\overline{\zeta}$. *Besides, the following representations are true*:

$$L_\omega \log|\widetilde{b}_\omega(z,\ \zeta)| = -\operatorname{Re}\int_0^\eta \frac{\omega(t)dt}{t - i(z-\zeta)} - \operatorname{Re}\int_0^\eta \frac{\omega(t)dt}{t + i(z-\overline{\zeta})}$$

$$= \operatorname{Re}\int_{-\eta}^\eta \frac{\omega(\eta - |t|)}{t + i(z-\xi)}\,dt, \quad z \notin [\zeta,\ \overline{\zeta}]. \tag{2.9}$$

Proof. It suffices to prove only the representations (2.9), since they easily imply the required harmonicity. For $y = \operatorname{Im} z > \eta$, the first line of (2.9) follows from formulas (1.1), (1.3), (2.1), and (2.2), due to absolute convergence of the integrals, and this representation is true for any $z \notin [\zeta,\ \overline{\zeta}]$ by the uniqueness of harmonic function. The second line of (2.9) follows from the first one by some simple change of variables.

Lemma 2.3. *For any* $\zeta = \xi + i\eta \in G^+$ *and* $z = x + iy \notin [\zeta,\ \overline{\zeta}]$,

$$L_\omega \log|\widetilde{b}_\omega(z,\ \zeta)| = -2y\int_0^\eta \frac{y^2 + (x-\xi)^2 - t^2}{[t^2 + y^2 + (x-\xi)^2]^2 - 4t^2y^2}\,\omega(\eta - t)dt, \tag{2.10}$$

$$L_\omega \log|\widetilde{b}_\omega(z,\ \zeta)| = \int_0^\eta \log|b_0(z,\ \zeta - i\sigma)|\,d\omega(\sigma). \tag{2.11}$$

Proof. Formula (2.10) easily follows from the first line of (2.9). Further, integrating by parts from the second line of (2.9), we get

$$L_\omega \log|\widetilde{b}_\omega(z,\ \zeta)| = \int_{-\eta}^\eta \omega(\eta - |t|)d\log[t + i(z-\xi)]$$

$$= \int_0^\eta \log[t + i(z-\xi)]\omega'(\eta - t)dt$$

$$- \int_{-\eta}^0 \log[t + i(z-\xi)]\omega'(\eta + t)dt.$$

Replacing $t \to -t$ in the last integral, we come to (2.11).

2.3. Observe that by the representation (2.10)

$$L_\omega \log|\widetilde{b}_\omega(x, \zeta)| = 0, \quad -\infty < x < +\infty, \quad x \neq \xi. \tag{2.12}$$

Besides, if $|z - \xi| \geq \eta,$ then the integrand in (2.10) is nonnegative. Hence,

$$L_\omega \log|\widetilde{b}_\omega(z, \zeta)| \begin{cases} < 0, & z \in G^+, \\ > 0, & z \in G^- = \{z : \operatorname{Im} z < 0\}, \end{cases}$$

when $|z - \xi| > \eta.$ For a further study of the function $L_\omega \log|\widetilde{b}_\omega(z, \zeta)|,$ the well-known properties of the Cauchy type integrals (see, e.g., [3], Chapter I) are to be used. Indeed, by formula (2.9), it easily follows that

$$L_\omega \log|\widetilde{b}_\omega(iz + \xi, \zeta)| = 2\pi \operatorname{Im} \Phi_\omega(z), \quad z \notin [-\eta, \eta], \tag{2.13}$$

where

$$\Phi_\omega(z) \equiv \frac{1}{2\pi i} \int_{-\eta}^{\eta} \frac{\omega(\eta - |t|)}{t - z}\, dt, \tag{2.14}$$

i.e., is a Cauchy type integral. Hence,

$$L_\omega \log|\widetilde{b}_\omega(-iz + \xi, \zeta)| = -L_\omega \log|\widetilde{b}_\omega(iz + \xi, \zeta)|, \quad z \in \mathbb{C}. \tag{2.15}$$

Obviously, the function $\Phi_\omega(z)$ is holomorphic everywhere in the finite complex plane $\mathbb{C},$ except the interval $[-\eta, \eta].$ Further, $\omega(x) \in \mathrm{Lip}_1$ in $[0, \Delta],$ due to the continuous differentiability of $\omega(x).$ Consequently, the following statements are true:

(a) The Cauchy type integral

$$\Phi_\omega(x) = \frac{1}{2\pi i} \int_{-\eta}^{\eta} \frac{\omega(\eta - |t|)}{t - x}\, dt,$$

in the sense of its principal value, is a continuous function on $[-\eta, \eta].$

(b) At any point $x \in (-\eta, \eta),$ the following limits exist and are finite:

$$\lim_{z \to x,\, z \in G^+} \Phi_\omega(z) \equiv \Phi_\omega^+(x), \qquad \lim_{z \to x,\, z \in G^-} \Phi_\omega(z) \equiv \Phi_\omega^-(x),$$

besides,

$$\Phi_\omega^+(x) - \Phi_\omega^-(x) = \omega(\eta - |x|), \qquad \Phi_\omega^+(x) + \Phi_\omega^-(x) = 2\Phi_\omega(x).$$

(c) For any $\lambda \in (0, 1)$, the limits $\Phi_\omega^+(x)$ and $\Phi_\omega^-(x)$ are continuous functions of the class Lip_1 in the interval $(-\eta, \eta)$.

(d) $\Phi_\omega(z)$ is continuous at the points $z = \pm\eta$.

Using these properties of $\Phi_\omega(z)$, we prove the following statement:

Theorem 2.2. *For any fixed* $\zeta = \xi + i\eta \in G^+$.

(i) *The function* $L_\omega \log|\widetilde{b}_\omega(z, \zeta)|$ *is continuous in the closed complex plane* $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, *is harmonic everywhere in the finite complex plane* $\mathbb{C}$, *except the closed straight line interval* $[\zeta, \overline{\zeta}]$ *with endpoints* ζ *and* $\overline{\zeta}$, *is subharmonic in the upper half-plane* G^+ *and superharmonic in the lower half-plane* G^-. *Besides,*

$$L_\omega \log|\widetilde{b}_\omega(z, \zeta)| \begin{cases} < 0, & z \in G^+, \\ > 0, & z \in G^-. \end{cases} \tag{2.16}$$

(ii) *The following equality is true*:

$$L_\omega \log|\widetilde{b}_\omega(x, \zeta)| = 0, \quad -\infty < x < +\infty. \tag{2.17}$$

(iii) *The following equality is true*:

$$\min_{z \in \overline{G^+}} L_\omega \log|\widetilde{b}_\omega(z, \zeta)| = -2\int_0^\eta \frac{\eta - x}{2\eta - x} \frac{\omega(x)}{x} dx, \quad \zeta = \xi + i\eta \in G^+. \tag{2.18}$$

Proof. (ii) By (2.13) and the properties of the Cauchy type integral (2.14), the function $L_\omega \log|\widetilde{b}_\omega(z, \zeta)|$ is continuous everywhere in $\mathbb{C}$ and vanishes at ∞. So, this function is continuous in the closed complex plane. The equality (2.17) follows from (2.12).

(i) The function $L_\omega \log|\widetilde{b}_\omega(z, \zeta)|$ is harmonic everywhere, except $[\zeta, \overline{\zeta}]$. Therefore, it suffices to prove that for any $s \in [\zeta, \overline{\zeta}]$ and any small enough number ρ,

$$\frac{1}{2\pi}\int_0^{2\pi} L_\omega \log|\widetilde{b}_\omega(s + \rho e^{i\vartheta}, \zeta)| d\vartheta - L_\omega \log|\widetilde{b}_\omega(s, \zeta)| \begin{cases} \geq 0, & s \in (\xi, \zeta], \\ \leq 0, & s \in [\overline{\zeta}, \xi). \end{cases}$$

Then, the inequalities (2.16) follow by the maximum principle of subharmonic functions in G^+ and minimum principle of superharmonic functions in G^-. For proving (2.16), observe that by (2.15) and (2.17) for $\rho > 0$,

$$\frac{1}{2\pi}\int_0^{2\pi} L_\omega \log|\widetilde{b}_\omega(\xi + \rho e^{i\vartheta}, \zeta)| d\vartheta - L_\omega \log|\widetilde{b}_\omega(\xi, \zeta)| = 0.$$

Further, suppose $0 < h < \eta$ and $0 < \rho < \max\{h, \eta - h\}$. Then, by (2.9),

$$\frac{1}{2\pi}\int_0^{2\pi} L_\omega \log|\widetilde{b}_\omega(\xi + ih + \rho e^{i\vartheta}, \zeta)| d\vartheta$$

$$= \operatorname{Re}\frac{1}{2\pi}\int_0^{2\pi} d\vartheta \int_{-\eta}^{\eta} \frac{\omega(\eta - |t|)}{t - h + i\rho e^{i\vartheta}} dt$$

$$= -\operatorname{Re}\int_{-\eta}^{\eta} \omega(\eta - |t|)\left(\frac{1}{2\pi}\int_{|s|=\rho} \frac{ds}{s[s - i(t - h)]}\right) dt.$$

Calculating the inner integral by residues, we get

$$\frac{1}{2\pi}\int_{|s|=\rho} \frac{ds}{s[s - i(t - h)]} = \begin{cases} -\dfrac{1}{t - h}, & |t - h| > \rho, \\ 0, & |t - h| < \rho. \end{cases}$$

Thus,

$$\frac{1}{2\pi}\int_0^{2\pi} L_\omega \log|\widetilde{b}_\omega(\xi + ih + \rho e^{i\vartheta}, \zeta)| d\vartheta = \left(\int_{-\eta}^{h-\rho} + \int_{h+\rho}^{\eta}\right) \frac{\omega(\eta - |t|)}{t - h} dt.$$

Consequently, if $0 < h < \eta$ and $0 < \rho < \max\{h, \eta - h\}$, then

$$\frac{1}{2\pi}\int_0^{2\pi} L_\omega \log|\widetilde{b}_\omega(\xi+ih+\rho e^{i\vartheta},\zeta)|d\vartheta - L_\omega \log|\widetilde{b}_\omega(\xi+ih,\zeta)|d\vartheta$$

$$= -\int_{h-\rho}^{h+\rho}\frac{\omega(\eta-|t|)}{t-h}\,dt = -\int_0^{\rho}\frac{\omega(\eta-x-h)}{x}\,dx - \int_{-\rho}^{0}\frac{\omega(\eta-|x+h|)}{x}\,dx$$

$$= \int_0^{\rho}\frac{\omega(\eta+x-h)-\omega(\eta-x-h)}{x}\,dx > 0,$$

since $\omega(x)$ is strictly increasing. For $h=\eta$, we assume that $0<\rho<\eta$ and get

$$\frac{1}{2\pi}\int_0^{2\pi} L_\omega \log|\widetilde{b}_\omega(\zeta+\rho e^{i\vartheta},\zeta)|d\vartheta$$

$$= \operatorname{Re}\frac{1}{2\pi}\int_0^{2\pi}d\vartheta\int_{-\eta}^{\eta}\frac{\omega(\eta-|t|)}{t+i\rho e^{i\vartheta}-\eta}\,dt$$

$$= -\operatorname{Re}\int_{-\eta}^{\eta}\left(\frac{1}{2\pi}\int_{|s|=\rho}\frac{ds}{s[s-i(t-\eta)]}\right)\omega(\eta-|t|)dt,$$

where

$$\frac{1}{2\pi}\int_{|s|=\rho}\frac{ds}{s[s-i(t-\eta)]} = \begin{cases} -\dfrac{1}{t-\eta}, & |\eta-t|>\rho,\\ 0, & |\eta-t|<\rho. \end{cases}$$

Thus,

$$\frac{1}{2\pi}\int_0^{2\pi} L_\omega \log|\widetilde{b}_\omega(\zeta+\rho e^{i\vartheta},\zeta)|d\vartheta = \int_{-\eta}^{\eta-\rho}\frac{\omega(\eta-|t|)}{t-\eta}\,dt,$$

and finally, for $0<\rho<\eta$,

$$\frac{1}{2\pi}\int_0^{2\pi} L_\omega \log|\widetilde{b}_\omega(\zeta+\rho e^{i\vartheta},\zeta)|d\vartheta - L_\omega \log|\widetilde{b}_\omega(\zeta,\zeta)|$$

$$= -\int_{\eta-\rho}^{\eta}\frac{\omega(\eta-|t|)}{t-\eta}\,dt = \int_0^{\rho}\frac{\omega(x)}{x}\,dx > 0.$$

The inequality in the second line of (2.16) follows from the already proved first line and (2.15).

(iii) The function $L_\omega \log|\widetilde{b}_\omega(z, \zeta)|$ $(\zeta = \xi + i\eta \in G^+)$ is nonpositive, continuous in $\overline{G^+}$, and harmonic in the domain $\overline{G^+} \setminus [\xi, \zeta]$. Besides, (2.17) is true. Hence, this function takes its minimal value on the closed interval $[\xi, \zeta]$. Namely,

$$\min_{z \in \overline{G^+}} L_\omega \log|\widetilde{b}_\omega(z, \zeta)| = L_\omega \log|\widetilde{b}_\omega(\zeta, \zeta)| = -\int_{-\eta}^{\eta} \frac{\omega(\eta - |t|)}{\eta - t}\, dt$$

$$= -2\int_0^{\eta} \frac{\eta - x}{2\eta - x} \frac{\omega(x)}{x}\, dx.$$

Below, we prove one more useful lemma:

Lemma 2.4. *For any fixed* $\zeta = \xi + i\eta \in G^+$,

$$\lim_{y \to 0} \int_{-\infty}^{+\infty} L_\omega \log|\widetilde{b}_\omega(x + iy, \zeta)|\, dx = 0. \tag{2.19}$$

Proof. By formula (2.9),

$$L_\omega \log|\widetilde{b}_\omega(x + iy, \zeta)| = \int_{-\eta}^{\eta} \frac{t - y}{(x - \xi)^2 + (t - y)^2}\, \omega(\eta - |t|)\, dt.$$

Hence, changing the order of integration, we get

$$\int_{-\infty}^{+\infty} L_\omega \log|\widetilde{b}_\omega(x + iy, \zeta)|\, dx = \int_{-\eta}^{\eta} \omega(\eta - |t|)\, dt \int_{-\infty}^{+\infty} \frac{t - y}{(x - \xi)^2 + (t - y)^2}\, dx$$

$$= \pi \int_{-\eta}^{\eta} \omega(\eta - |t|) \operatorname{sign}(t - y)\, dt \equiv D_\omega(\eta, y).$$

For calculating the last integral, observe that if $|y| \geq \eta$, then

$$D_\omega(\eta, y) = -2\pi(\operatorname{sign} y) \int_0^{\eta} \omega(x)\, dx,$$

and if $|y| < \eta$, then it is easy to verify that

$$D_\omega(\eta, y) = -2\pi(\operatorname{sign} y) \int_{\eta - y}^{\eta} \omega(x)\, dx.$$

Thus,

$$\int_{-\infty}^{+\infty} L_\omega \log|\widetilde{b}_\omega(x+iy,\ \zeta)|dt = \begin{cases} -2\pi(\operatorname{sign} y)\int_0^\eta \omega(x)dx, & |y| \geq \eta, \\ -2\pi(\operatorname{sign} y)\int_{\eta-|y|}^{\eta} \omega(x)dx, & |y| < \eta. \end{cases} \tag{2.20}$$

Hence (2.19) follows.

3. Green Type Potentials

The theorems of this section relate to the convergence and some properties of the Green type potentials constructed by means of Blaschke type factors of the previous section.

3.1. First, we prove the following theorem:

Theorem 3.1. *If a nonnegative Borel measure* $\nu(\zeta)$, *given in the half-plane* G^+, *satisfies the condition*

$$\iint_{G^+}\left(\int_0^{\operatorname{Im}\zeta} \omega(t)dt\right)d\nu(\zeta) < +\infty, \tag{3.1}$$

then the Green type potential

$$\widetilde{P}_\omega(z) = \iint_{G^+} \log|\widetilde{b}_\omega(z,\ \zeta)|d\nu(\zeta), \tag{3.2}$$

is convergent in G^+ *and represents there a subharmonic function with the Riesz measure* $\nu(\zeta)$.

Proof. In any half-plane $G_\rho^+ = \{z : \operatorname{Im} z > \rho\}$ with $0 < \rho < \Delta$, we define a Green type potential as the sum

$$\widetilde{P}_\omega(z) = P_0(z,\ \rho) + U_\omega(z,\ \rho), \quad z \in G_\rho^+, \tag{3.3}$$

where

$$P_0(z) \equiv \iint_{G_\rho^+} \log\left|\frac{z-\zeta}{z-\bar{\zeta}}\right| d\nu(\zeta) = \iint_{G_\rho^+} \log|b_0(z, \zeta)| d\nu(\zeta), \tag{3.4}$$

is an ordinary Green potential, and

$$U_\omega(z, \rho) \equiv \iint_{G^+ \setminus G_\rho^+} \log|\tilde{b}_\omega(z, \zeta)| d\nu(\zeta) + \iint_{G_\rho^+} \log\left|\frac{\tilde{b}_\omega(z, \zeta)}{b_0(z, \zeta)}\right| d\nu(\zeta)$$

$$\equiv U_\omega^{(1)}(z, \rho) + U_\omega^{(2)}(z, \rho). \tag{3.5}$$

The Green type potential $\tilde{P}_\omega(z)$ is convergent in G^+ in the sense that for any $\rho \in (0, \Delta)$, the ordinary Green potential $P_0(z, \rho)$ converges in G^+ and $U_\omega(z, \rho)$ is a harmonic function in G_ρ^+. For verifying this, we first prove that if the condition (3.1) is fulfilled, then for any $\rho \in (0, \Delta)$,

$$\iint_{G_\rho^+} \operatorname{Im} \zeta d\nu(\zeta) < +\infty, \quad \zeta = \xi + i\eta.$$

Indeed, if $0 < \rho < \Delta$, then by (3.1),

$$+\infty > \iint_{G_\rho^+} \left(\int_0^\eta \omega(t) dt\right) d\nu(\zeta)$$

$$= \left(\iint_{G_\Delta^+} + \iint_{G_\rho^+ \setminus G_\Delta^+}\right) \left(\int_0^\eta \omega(t) dt\right) d\nu(\zeta)$$

$$\geq \iint_{G_\Delta^+} \left(\int_0^\Delta \omega(t) dt + \int_\Delta^\eta \omega(t) dt\right) d\nu(\zeta) + \iint_{G_\rho^+ \setminus G_\Delta^+} \left(\int_0^\rho \omega(t) dt\right) d\nu(\zeta)$$

$$= \iint_{G_\Delta^+} [C_1 + C_2(\eta - \Delta)] d\nu(\zeta) + C_3 \iint_{G_\rho^+ \setminus G_\Delta^+} d\nu(\zeta)$$

$$\geq C_4 \iint_{G_\Delta^+} \eta d\nu(\zeta),$$

where $C_{1,\,2,\,3,\,4} > 0$ are some constants. Now, observe that the already proved relation (3.5) provides the convergence of the ordinary Green potential in (3.4) for any $\rho \in (0, \Delta)$, since it is well-known that its convergence is guaranteed even by the weaker Blaschke condition

$$\iint_{G_\rho^+} \frac{\operatorname{Im}\zeta}{1+|\zeta|^2}\, d\nu(\zeta) < +\infty.$$

Further, by Theorem 2.1, the integrand $\log|\widetilde{b}_\omega(z, \zeta)|$ in $U_\omega^{(1)}(z, \rho)$ is harmonic in G_ρ^+. Besides, assuming that $z = x + iy, \zeta = \xi + i\eta$, and $y < \rho_1$, where $\rho_1 > \rho$ is fixed, one can be convinced that by (2.1) and (1.3),

$$\begin{aligned}\left|\log|\widetilde{b}_\omega(z, \zeta)|\right| &= \left|\int_0^\eta \{C_\omega(z - \zeta + it) + C_\omega(z - \overline{\zeta} - it)\}\omega(t)dt\right| \\ &\le \int_0^\eta \{C_\omega(i(y - \eta + t)) + C_\omega(i(y + \eta - t))\}\omega(t)dt \\ &\le 2C_\omega(i(\rho_1 - \rho))\int_0^\eta \omega(t)dt,\end{aligned}$$

and hence,

$$\begin{aligned}\left|U_\omega^{(1)}(z, \rho)\right| &\le \iint_{G^+\setminus G_\rho^+} \left|\log|\widetilde{b}_\omega(z, \zeta)|\right| d\nu(\zeta) \\ &\le 2C_\omega(i(\rho_1 - \rho))\iint_{G^+\setminus G_\rho^+}\left(\int_0^\eta \omega(t)dt\right)d\nu(\zeta) < +\infty.\end{aligned}$$

Thus, the modulus of the integrand in $U_\omega^{(1)}(z, \rho)$, which is a harmonic function in G_ρ^+, possesses an independent of $z \in G_{\rho_1}^+$, integrable majorant. Hence, $U_\omega^{(1)}(z, \rho)$ is a harmonic function in $z \in G_\rho^+$.

For proving that $U_\omega^{(2)}(z, \rho)$ is harmonic in the whole G^+, observe that by the representation (2.4) and (2.5)

$$\log\left|\frac{\widetilde{b}_\omega(z, \zeta)}{b_0(z, \zeta)}\right| = -\operatorname{Re} J_\omega(z, \zeta), \quad z \in G^+,$$

where $J_\omega(z, \zeta)$ is a holomorphic function in G^+. Besides, by (2.8) for any $y > \varepsilon > 0$ and $\delta \in (0, 1)$,

$$|J_\omega(z, \zeta)| \leq 2 \frac{M_{\varepsilon,\delta}}{\varepsilon^{1-\delta}} \left[\eta \int_0^{+\infty} d\omega(t) + \int_0^\eta t d\omega(t) \right] \leq 4 \frac{M_{\varepsilon,\delta}}{\varepsilon^{1-\delta}} \omega(\Delta)\eta.$$

Consequently, for any $y > \varepsilon$ and $\delta \in (0, 1)$,

$$\iint_{G_\rho^+} |J_\omega(z, \zeta)| d\nu(\zeta) \leq 4 \frac{M_{\varepsilon,\delta}}{\varepsilon^{1-\delta}} \omega(\Delta) \iint_{G_\rho^+} \eta d\nu(\zeta) < +\infty.$$

Thus, for any $\varepsilon > 0$, the modulus of the integrand in $U_\omega^{(2)}(z, \rho)$, which is a harmonic function in G_ρ^+, possesses an integrable majorant independent of $z \in G_\varepsilon^+$. Consequently, $U_\omega^{(2)}(z, \rho)$ is a harmonic function in G^+.

3.2. The next two theorems relate to some properties of the potential $\widetilde{P}_\omega(z)$, which are revealed by application of the operator L_ω.

Theorem 3.2. *If a nonnegative Borel measure* $\nu(\zeta)$ *given in* G^+ *satisfies the condition* (3.1), *then* $L_\omega \widetilde{P}_\omega(z)$ *is a nonpositive, continuous, subharmonic function in* G^+, *and*

$$L_\omega \widetilde{P}_\omega(z) = \iint_{G^+} L_\omega \log|\widetilde{b}_\omega(z, \zeta)| d\nu(\zeta), \quad z \in G^+, \tag{3.6}$$

where the integral is absolutely and uniformly convergent inside G^+.

Proof. Assuming that $\mathfrak{K} \subset G^+$ is any compact with $d = \min_{z \in \mathfrak{K}} \operatorname{Im} z > 0$ and $0 < \rho < \min\{d / 2, \Delta\}$ is a fixed number, we write

$$\widetilde{P}_\omega(z) = \iint_{G^+ \setminus G_\rho^+} \log|\widetilde{b}_\omega(z, \zeta)| d\nu(\zeta) + \iint_{G_\rho^+} \log|\widetilde{b}_\omega(z, \zeta)| d\nu(\zeta)$$

$$\equiv P_\omega^{(1)}(z) + P_\omega^{(2)}(z),$$

and separately prove the desired statements for $P_\omega^{(1)}(z)$ and $P_\omega^{(2)}(z)$. To this end, it suffices to show that the integral in the right-hand side of formula (3.6), written for $P_\omega^{(1)}(z)$ and $P_\omega^{(2)}(z)$, is absolutely and uniformly convergent with respect to $z \in \mathfrak{K}$. Then, formula (3.6) follows by Fubini's theorem, while the subharmonicity and the continuity of the function $L_\omega \widetilde{P}_\omega(z)$ will hold by the same properties of $L_\omega \log|\widetilde{b}_\omega(z, \zeta)|$.

For $P_\omega^{(1)}(z)$, observe that by (2.1) and (1.2) for $d < y < +\infty$ and $0 < \eta < \operatorname{Im}\zeta < \rho < d/2$,

$$|\log|\widetilde{b}_\omega(z, \zeta)|| \le 2\int_0^\eta \omega(t)dt \int_0^{+\infty} e^{-\sigma(y-\eta+t)} \frac{d\sigma}{I_\omega(\sigma)}$$

$$< 2C_\omega(id/2)\int_0^\eta \omega(t)dt < +\infty.$$

Hence, the desired statement follows by (3.1). Proceeding to estimation of the integrand in $P_\omega^{(2)}(z)$, where $\eta > \rho$ and $z = x + iy \in \mathfrak{K}$, i.e., $y \ge d > 2\rho$, observe that by (2.4) $\log|\widetilde{b}_\omega(z, \zeta)| = \log|b_0(z, \zeta)| - \operatorname{Re} J_\omega(z, \zeta)$, and hence

$$|\log|\widetilde{b}_\omega(z, \zeta)|| \le |\log|b_0(z, \zeta)|| + |J_\omega(z, \zeta)|. \tag{3.7}$$

Besides,

$$L_\omega|\log|b_0(z, \zeta)|| \le |L_\omega \log|\widetilde{b}_\omega(z, \zeta)|| + |L_\omega J_\omega(z, \zeta)|, \tag{3.8}$$

where by (2.18),

$$|L_\omega \log|\widetilde{b}_\omega(z, \zeta)|| \le 2\int_0^\Delta \frac{\eta - x}{2\eta - x} \frac{\omega(x)}{x} dx \le M_\rho \int_0^\eta \omega(x)dx,$$

with some constant $M_\rho > 0$ depending solely on ρ. On the other hand, from (2.5) and (2.7), it follows that

$$|J_\omega(z, \zeta)| \leq \frac{C_\omega(id)}{\pi}\int_0^{+\infty} d\omega(t)\int_{-\infty}^{+\infty} |\log|b_0(u + it)||du$$

$$= 2C_\omega(id)\left\{\int_0^\eta t d\omega(t) + \eta\int_\eta^\Delta d\omega(t)\right\},$$

where the last integral in the figure brackets disappears for $\eta > \Delta$. Hence, we conclude that

$$|J_\omega(z, \zeta)| \leq 2C_\omega(id)\Delta\omega(\Delta) \leq M_{d,\rho}\int_0^\eta \omega(x)dx,$$

and

$$L_\omega|J_\omega(z, \zeta)| \leq M_{d,\rho}\omega(\Delta)\int_0^\eta \omega(x)dx \equiv M'_{d,\rho}\int_0^\eta \omega(x)dx,$$

where $M_{d,\rho}$ and $M'_{d,\rho}$ are some positive constants depending only on d and ρ. By the above estimates and formulas (3.7) and (3.8), we conclude that in $P_\omega^{(2)}(z)$

$$L_\omega|\log|\widetilde{b}_\omega(z, \zeta)|| \leq L_\omega|\log|b_0(z, \zeta)|| + L_\omega|J_\omega(z, \zeta)|$$

$$\leq (M_\rho + M'_{d,\rho} + M_{d,\rho})\int_0^\eta \omega(x)dx,$$

and again the desired statement follows by (3.1).

Theorem 3.3. *If a given in* G^+ *nonnegative Borel measure* $\nu(\zeta)$ *satisfies the condition* (3.1), *then*

$$\sup_{y>0}\int_{-\infty}^{+\infty}\left|L_\omega\widetilde{P}_\omega(x + iy)\right|dx < +\infty, \tag{3.9}$$

$$\lim_{y\to+0}\int_{-\infty}^{+\infty}\left|L_\omega\widetilde{P}_\omega(x + iy)\right|dx = 0. \tag{3.10}$$

Proof. Observe that in view of formulas (3.6), (2.20), and the nonpositivity of $L_\omega \log|\widetilde{b}_\omega(z, \zeta)|$ in G^+

$$\int_{-\infty}^{+\infty} |L_\omega \widetilde{P}_\omega(x + iy)| dx$$

$$= \int_{-\infty}^{+\infty} dx \iint_{G^+} \left| L_\omega \log|\widetilde{b}_\omega(x + iy, \zeta)| \right| d\nu(\zeta)$$

$$= \iint_{G^+} d\nu(\zeta) \int_{-\infty}^{+\infty} \left| L_\omega \log|\widetilde{b}_\omega(x + iy, \zeta)| \right| dx$$

$$= 2\pi \iint_{G_y^+} \left(\int_{\eta - y}^{\eta} \omega(x) dx \right) d\nu(\zeta) + 2\pi \iint_{G^+ \setminus G_y^+} \left(\int_0^{\eta} \omega(x) dx \right) d\nu(\zeta)$$

$$\equiv 2\pi (A(y) + B(y)).$$

Hence, we easily come to the relation (3.9)

$$\sup_{y > 0} \int_{-\infty}^{+\infty} \left| L_\omega \widetilde{P}_\omega(x + iy) \right| dx \leq 4\pi \iint_{G^+} \left(\int_0^{\eta} \omega(x) dx \right) d\nu(\zeta) < +\infty.$$

Further, it is obvious that $B(y) \to 0$ as $y \to +0$. As to $A(y)$, its integrand $\int_{\eta - y}^{\eta} \omega(x) dx$ possesses an independent of y integrable majorant $\int_0^{\eta} \omega(x) dx$. Therefore, denoting the characteristic function of the half-plane G_y^+ by $\chi_y(\zeta)$, we get

$$\lim_{y \to +0} A(y) = \iint_{G^+} \lim_{y \to +0} \left(\chi_y(\zeta) \int_{\eta - y}^{\eta} \omega(x) dx \right) d\nu(\zeta) = 0.$$

Remark 3.1. As we have proved, under the condition (3.2), the function $L_\omega \widetilde{P}_\omega(z)$ is subharmonic in G^+ and the relation (3.9) is true. This means that $L_\omega \widetilde{P}_\omega(z)$ belongs to the class $\mathfrak{N}^m$ of Solomentsev [10]. On the other hand, the relation (3.10) provides the equality of $L_\omega \widetilde{P}_\omega(z)$ to some ordinary Green potential, which possesses the minimality property, i.e., at almost all points of the real axis, it has zero boundary limits by the orthogonal to the real axis arcs of any circle, which is the image of a radius under a conformal mapping of the disc $|z| < 1$ to the half-plane G^+.

4. One More Property of Green Type Potentials

In this section, we prove one more property of the Green type potential $\widetilde{P}_\omega(z)$. To this end, first we prove the following improvement of the estimate of the kernel (1.2) given by Lemma 3.2 in [7].

Lemma 4.1. *For any fixed number* $\rho > 0$,

$$|\operatorname{Re} C_\omega(z)| \leq M_{\omega, \rho} \frac{1 + y}{|z|^2}, \quad z = x + iy \in G_\rho^+, \tag{4.1}$$

where $M_{\omega, \rho} > 0$ *is a constant depending only on* ρ *and the function* $\omega(x)$.

Proof. For any $x + iy \in G_\rho^+$, integration by parts gives

$$C_\omega(z) = \int_0^{+\infty} e^{itz} \frac{dt}{I_\omega(t)} = \frac{1}{iz} \frac{e^{itz}}{I_\omega(t)}\Bigg|_{t=0}^{+\infty} - \frac{1}{iz} \int_0^{+\infty} e^{itz} \frac{I'_\omega(t)}{[I_\omega(t)]^2} dt$$

$$= \frac{1}{iz} \frac{e^{itz}}{I_\omega(t)}\Bigg|_{t=0}^{+\infty} - \frac{1}{(iz)^2} e^{itz} \frac{I'_\omega(t)}{[I_\omega(t)]^2}\Bigg|_{t=0}^{+\infty}$$

$$+ \frac{1}{(iz)^2} \int_0^{+\infty} e^{itz} \left\{ \frac{I''_\omega(t)}{[I_\omega(t)]^2} - 2 \frac{I'_\omega(t)}{[I_\omega(t)]^3} \right\} dt. \tag{4.2}$$

Observe that for $0 \leq t < +\infty$,

$$I_\omega(t) = \int_0^{+\infty} e^{-tx} d\omega(x),$$

is a positive, continuous function and $I_\omega(0) = \int_0^{+\infty} d\omega(x) = \omega(\Delta)$. Further, choose a number $\delta \in (0, \min\{\rho/3, \Delta/3\})$ and denote $C_1 \equiv \min_{\delta \le x \le \Delta/2} \omega'(x) > 0$. Then, for t large enough,

$$I_\omega(t) \ge \int_\delta^{\Delta/2} e^{-tx} d\omega(x) \ge C_1 \int_\delta^{\Delta/2} e^{-tx} dx = \frac{C_1}{t} e^{-t\delta}\left(1 - e^{-t(\Delta/2-\delta)}\right) > \frac{C_1}{2t} e^{-t\delta}.$$

Consequently,

$$I_\omega(t) \ge \frac{C_2}{1+t} e^{-t\delta}, \quad 0 \le t < +\infty, \tag{4.3}$$

where $G_2 > 0$ is some constant. Further, for any $0 \le t < +\infty$, the function

$$\frac{I'_\omega(t)}{[I_\omega(t)]^2} = -\frac{\int_0^\Delta e^{-tx} x d\omega(x)}{\left[\int_0^\Delta e^{-tx} d\omega(x)\right]^2},$$

is negative and continuous, and it is easy to see that for t large enough

$$\left|\frac{I'_\omega(t)}{[I_\omega(t)]^2}\right| \le C_3 t^2 e^{2t\delta},$$

where $C_3 > 0$ is some constant. Consequently,

$$\left|\frac{I'_\omega(t)}{[I_\omega(t)]^2}\right| \le C_4 (1+t)^2 e^{2t\delta}, \quad 0 \le t < +\infty, \tag{4.4}$$

with some constant $C_4 > 0$. In a similar way, we come also to the estimate

$$\left|\frac{I''_\omega(t)}{[I_\omega(t)]^2} - 2\frac{I'_\omega(t)}{[I_\omega(t)]^3}\right| \le C_5 (1+t)^3 e^{3t\delta}, \quad 0 \le t < +\infty, \tag{4.5}$$

where $C_5 > 0$ is some constant.

It follows from the representation (4.2) and the estimates (4.3), (4.4), and (4.5) that

$$\operatorname{Re} C_{\omega}(z) = \frac{1}{\omega(\Delta)} \frac{y}{|z|^2} + \operatorname{Re} \psi(z), \quad z \in G_{\rho}^{+}, \tag{4.6}$$

where the function

$$\psi(z) = -\frac{1}{(iz)^2} \frac{1}{[\omega(\Delta)]^2} \int_0^{\Delta} x d\omega(x) + \frac{1}{(iz)^2} \int_0^{+\infty} e^{itz} \left\{ \frac{I''_{\omega}(t)}{[I_{\omega}(t)]^2} - 2 \frac{I'_{\omega}(t)}{[I_{\omega}(t)]^3} \right\} dt,$$

admits the estimate

$$|\psi(z)| \leq \frac{C_6}{|z|^2} + \frac{C_5}{|z|^2} \int_0^{+\infty} e^{-t(\rho - 3\delta)} (1 + t)^3 dt \equiv \frac{C_7}{|z|^2}, \quad z \in G_{\rho}^{+},$$

with some constants $C_{6,\,7} > 0.$ Hence, the desired estimate (4.1) holds by (4.6).

Along with Lemma 4.1, we shall use the following statement on the Green type potentials:

Theorem 4.1. *If a nonnegative Borel measure* $\nu(\zeta)$ *satisfies the condition* (3.1), *then for any* $\rho > 0,$ *the corresponding Green type potential satisfies the condition*

$$\sup_{y > \rho} \int_{-\infty}^{+\infty} |\widetilde{P}_{\omega}(x + iy)| dx < +\infty. \tag{4.7}$$

Proof. Assuming that $0 < \rho / 2 < \Delta,$ we represent the Green type potential in the half-plane $G_{\rho/2}^{+}$ in the form (3.3), i.e., as the sum of integrals $P_0(z, \rho / 2)$ and $U_{\omega}(z, \rho / 2),$ and estimate these integrals separately. To this end, first we observe that the condition (3.1) implies that

$$\iint_{G_{\rho}^{+}} \operatorname{Im} \zeta d\nu(\zeta) < +\infty, \quad \zeta = \xi + i\eta.$$

Therefore, by (2.7),

$$\sup_{y>\rho/2}\int_{-\infty}^{+\infty}|P_0(x+iy,\ \rho/2)|dx \le 2\pi\iint_{G_\rho^+/2}\eta d\nu(\zeta) < +\infty \qquad (\zeta = \xi + i\eta).$$

Further, we represent $U_\omega(z,\ \rho/2)$ in $G^+_{\rho/2}$ as the sum of integrals $U_\omega^{(1)}(z,\ \rho/2)$ and $U_\omega^{(2)}(z,\ \rho/2)$, as in (3.5). Then, by (2.1) and the estimate (4.1), we obtain that for $0 < \eta < \rho/2$ and $y > \rho$

$$\begin{aligned}
&\left|\log|\tilde{b}_\omega(z,\ \zeta)|\right| \\
&\quad \le \int_0^\eta \{|\operatorname{Re} C_\omega(z-\zeta+it)| + |\operatorname{Re} C_\omega(z-\overline{\zeta}-it)|\}\omega(t)dt \\
&\quad \le M_1\int_0^\eta\left\{\frac{1+y-\eta+t}{(x-\xi)^2+(y-\eta+t)^2}+\frac{1+y+\eta-t}{(x-\xi)^2+(y+\eta-t)^2}\right\}\omega(t)dt \\
&\quad \le M_2\int_0^\eta \frac{y}{(x-\xi)^2+M_3y^2}\,\omega(t)dt,
\end{aligned}$$

where $M_{1,\ 2,\ 3} > 0$ are some constants. Consequently,

$$\begin{aligned}
&\sup_{y>\rho}\int_{-\infty}^{+\infty}\left|U_\omega^{(1)}(x+iy,\ \rho/2)\right|dx \\
&\qquad \le M_2\sup_{y>\rho}\iint_{G^+\setminus G^+_{\rho/2}} d\nu(\zeta)\int_{-\infty}^{+\infty}\frac{ydx}{(x-\xi)^2+M_3y^2}\int_0^\eta \omega(t)dt \\
&\qquad \le \frac{\pi M_2}{\sqrt{M_3}}\iint_{G^+\setminus G^+_{\rho/2}}\left(\int_0^\eta \omega(t)dt\right)d\nu(\zeta) < +\infty.
\end{aligned}$$

Further, by the representation (2.4),

$$U_\omega^{(2)}(z,\ \rho/2) = -\iint_{G^+_{\rho/2}} \operatorname{Re} J_\omega(z,\ \zeta)d\nu(\zeta),$$

where $J_\omega(z,\ \zeta)$ is the integral (2.5). Further, again by the estimate (4.1) for $y > \rho$, we get

$$|\operatorname{Re} J_\omega(z, \zeta)| \leq M_4 \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{|L_\omega \log|b_0(u, \zeta)||}{|z - u|^2} du$$

$$= M_4 \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{du}{(x - u)^2 + y^2} \int_0^\Delta |\log|b_0(u + it, \zeta)|| d\omega(t),$$

where $M_4 > 0$ is some constant. Consequently, by formula (2.7) for any $y > \rho$ and $\eta > \rho / 2$,

$$\int_{-\infty}^{+\infty} |J_\omega(x + iy, \zeta)| dx$$

$$\leq M_5 \int_0^\Delta d\omega(t) \int_{-\infty}^{+\infty} |\log|b_0(u + it, \zeta)|| du \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{dx}{(x - u)^2 + y^2}$$

$$= 2\pi M_5 \left[\eta \int_\eta^{+\infty} d\omega(t) + \int_0^\eta t d\omega(t) \right] \leq M_6 \eta,$$

where $M_{5, 6} > 0$ are some constants. Consequently,

$$\sup_{y > \rho} \int_{-\infty}^{+\infty} \left| U_\omega^{(2)}(x + iy, \rho / 2) \right| dx \leq M_7 \iint_{G_{\rho/2}^+} \eta d\nu(\zeta) < +\infty,$$

for some constant $M_7 > 0$.

5. Representations of Classes of Harmonic Functions

We start by the following theorem on representations of some weighted classes of harmonic functions in G^+.

Theorem 5.1. (1°) *The class of harmonic in* G^+ *functions* $U(z)$, *for which*

$$\sup_{y > \rho} \int_{-\infty}^{+\infty} |U(x + iy)| dx < +\infty, \tag{5.1}$$

for any $\rho > 0$ *and*

$$\sup_{y>0} \int_{-\infty}^{+\infty} |L_\omega U(x+iy)| dx < +\infty, \tag{5.2}$$

coincides with the set of functions representable in the form

$$U(z) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \operatorname{Re} C_\omega(z-t) d\mu(t), \quad z \in G^+, \tag{5.3}$$

where $\mu(t)$ *is a function of bounded variation on* $(-\infty, +\infty)$.

(2°) *If the representation* (5.3) *is true, then the following analogue of the Stieltjes inversion formula holds*:

$$\mu(t) = \lim_{y \to +0} \int_0^x L_\omega U(t+iy) dt \quad \text{a.e.} \quad x \in (-\infty, +\infty). \tag{5.4}$$

Proof. (1°) First, let us verify that if $U(z)$ is harmonic in G^+, then also $L_\omega U(z)$ is harmonic in G^+. Indeed, $U(z)$ is uniformly continuous in any compact inside G^+. Hence, if $z = x + iy \in G^+$, then for any number $\varepsilon > 0$, there is some $\delta \in (0, y)$ such that

$$|U(z+i\sigma) - U(z+i\sigma+\rho e^{i\vartheta})| < \frac{\varepsilon}{\omega(\Delta)},$$

for any $0 \le \sigma \le \Delta$, provided $0 < \rho < \delta$. Consequently,

$$|L_\omega U(z) - L_\omega U(z+\rho e^{i\vartheta})| \le \int_0^\Delta |U(z+i\sigma) - U(z+i\sigma+\rho e^{i\vartheta})| d\omega(\sigma) < \varepsilon,$$

when $\rho > 0$ is small enough. Besides, it is easy to see that

$$\frac{1}{2\pi} \int_0^{2\pi} L_\omega U(z+\rho e^{i\vartheta}) d\vartheta = \frac{1}{2\pi} \int_0^{2\pi} d\vartheta \int_0^\Delta U(z+i\sigma+\rho e^{i\vartheta}) d\omega(\sigma)$$

$$= \int_0^\Delta d\omega(\sigma) \frac{1}{2\pi} \int_0^{2\pi} U(z+i\sigma+\rho e^{i\vartheta}) d\vartheta$$

$$= \int_0^\Delta U(z+i\sigma) d\omega(\sigma) = L_\omega U(z).$$

Now, suppose that a harmonic in G^+ function $U(z)$ is such that the relations (5.1) and (5.2) are true. Then, it is well-known (see, e.g., [6], Lemma 1.3 on p. 48) that (5.1) implies

$$U(z) = \frac{y-\rho}{\pi}\int_{-\infty}^{+\infty}\frac{U(t+i\rho)dt}{(x-t)^2+(y-\rho)^2}, \quad z = x+iy \in G_\rho^+, \tag{5.5}$$

for any $\rho > 0$. It follows from this representation, that for any $\rho > 0$, the function $U(z)$ is the real part of some function $f_\rho(z)$, which is holomorphic in G_ρ^+ and can be written as a Laplace transform. Indeed,

$$\begin{aligned} U(z) &= \operatorname{Re}\left\{\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{U(t+i\rho)dt}{-i(z-i\rho-t)}\right\} \\ &= \operatorname{Re}\left\{\frac{1}{\pi}\int_{-\infty}^{+\infty}U(t+i\rho)dt\int_0^{+\infty}e^{i\tau(z-i\rho-t)}d\tau\right\} \\ &= \operatorname{Re}\left\{\int_0^{+\infty}e^{i\tau(z-i\rho)}\left[\frac{1}{\pi}\int_{-\infty}^{+\infty}e^{-i\tau t}U(t+i\rho)dt\right]d\tau\right\} \\ &\equiv \operatorname{Re}\{f_\rho(z)\}, \quad z \in G_\rho^+, \end{aligned} \tag{5.6}$$

where all integrals are absolutely and uniformly convergent inside G_ρ^+. Hence,

$$L_\omega U(z) = \operatorname{Re}\{L_\omega f_\rho(z)\}, \quad z \in G_\rho^+,$$

where $L_\omega f_\rho(z)$ is a holomorphic in G_ρ^+ function representable as a Laplace transform. Indeed,

$$\begin{aligned} L_\omega f_\rho(z) &= \int_0^{+\infty} f_\rho(z+i\sigma)d\omega(\sigma) \\ &= \int_0^{+\infty}d\omega(\sigma)\int_0^{+\infty}e^{i\tau(z-i\rho+i\sigma)}\left\{\frac{1}{\pi}\int_{-\infty}^{+\infty}e^{-i\tau t}U(t+i\rho)dt\right\}d\tau \\ &= \int_0^{+\infty}e^{i\tau(z-i\rho)}\left\{\frac{I_\omega(\tau)}{\pi}\int_{-\infty}^{+\infty}e^{-i\tau t}U(t+i\rho)dt\right\}d\tau, \end{aligned}$$

where all integrals are absolutely and uniformly convergent inside G^+_ρ, including $I_\omega(\tau)$ defined in (1.2). Further, the condition (5.2), which is true for the function $L_\omega U(z)$ harmonic in G^+, implies the representation

$$L_\omega U(z) = y\pi\pi \int_{-\infty}^{+\infty} \frac{d\mu(t)}{(x-t)^2 + y^2}, \quad z = x + iy \in G^+, \tag{5.7}$$

where $\mu(t)$ is a function of bounded variation on $(-\infty, +\infty)$. Hence, the function $L_\omega U(z)$ is the real part of some Laplace transform, namely, of the holomorphic in G^+ function

$$F(z) = \int_0^{+\infty} e^{i\tau z} \left\{ \frac{1}{\pi} \int_{-\infty}^{+\infty} e^{-i\tau t} d\mu(t) \right\} d\tau, \quad z \in G^+,$$

and

$$L_\omega U(z) = \text{Re}\, F(z), \quad z \in G^+.$$

By (5.6), $\text{Re}\, F(z) = \text{Re}\, L_\omega f_\rho(z) (z \in G^+_\rho)$ for any $\rho > 0$, and hence

$$F(z) = L_\omega f_\rho(z) + iC_\rho, \quad z \in G^+_\rho,$$

where C_ρ is a real constant depending on ρ. But for any $\rho > 0$,

$$\lim_{y \to +\infty} F(iy) = \lim_{y \to +\infty} L_\omega f_\rho(iy) = 0,$$

since the generating functions of the Laplace transforms representing $F(z)$ and $f_\rho(z)$ are bounded. Thus, $C_\rho = 0$ for any $\rho > 0$.

So, for any $\rho > 0$, the function $L_\omega f_\rho(z)$ has a holomorphic continuation to the whole half-plane G^+, where

$$F(z) \equiv L_\omega f_\rho(z), \quad z \in G^+.$$

Consequently, by the uniqueness of the generating functions of Laplace transforms (see, e.g., [11], Chapter II, Section 6),

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} e^{-i\tau t} d\mu(t) = \frac{I_\omega(\tau)}{\pi}\int_{-\infty}^{+\infty} e^{-i\tau t} U(t + i\rho) dt, \quad 0 < \tau < +\infty,$$

where the right-hand side does not depend on $\rho > 0$. Hence, for any $\rho > 0$,

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} e^{-i\tau t} U(t + i\rho) dt = \frac{1}{\pi I_\omega(\tau)}\int_{-\infty}^{+\infty} e^{-i\tau t} d\mu(t), \quad 0 < \tau < +\infty,$$

and coming to the Laplace transforms of these functions, by (5.6), we conclude

$$\begin{aligned} f_\rho(z) &= \int_0^{+\infty} e^{i\tau(z - i\rho)}\left\{\frac{1}{\pi}\int_{-\infty}^{+\infty} e^{-i\tau t} U(t + i\rho) dt\right\} d\tau \\ &= \int_0^{+\infty} e^{i\tau(z - i\rho)}\left\{\frac{1}{\pi I_\omega(\tau)}\int_{-\infty}^{+\infty} e^{-i\tau t} d\mu(t)\right\} d\tau \\ &= \frac{1}{\pi}\int_{-\infty}^{+\infty}\left\{\int_0^{+\infty} e^{i\tau(z - i\rho - t)}\, \frac{d\tau}{I_\omega(\tau)}\right\} d\mu(t) \\ &= \frac{1}{\pi}\int_{-\infty}^{+\infty} C_\omega(z - i\rho - t) d\mu(t), \quad z \in G_\rho^+. \end{aligned}$$

Consequently, for any $\rho > 0$,

$$U(z) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \operatorname{Re} C_\omega(z - i\rho - t) d\mu(t), \quad z \in G_\rho^+,$$

and letting $\rho \to +\infty$, we obtain the representation (5.3).

Conversely, let the representation (5.3) be true. Then, by the estimate (4.1), it easily follows that $U(z)$ is harmonic in G^+ and the relation (5.1) is true. As to the relation (5.2), it follows by application of the operator L_ω to both sides of formula (5.3), which gives (5.7).

(2°) The statement is obvious in virtue of formula (5.7).

6. Riesz Type Representations with Minimality Property

Henceforth, we assume that $U(z)$ is a δ-subharmonic function in the upper half-plane G^+, and its Riesz associated measure $\nu(\zeta)$ is minimally decomposed in the Jordan sense, i.e., $\nu(\zeta) = \nu_+(\zeta) - \nu_-(\zeta)$, where $\nu_\pm(\zeta)$ are the positive and the negative variations of the measure $\nu(\zeta)$, which are some nonnegative Borel measures with non-overlapping supports in G^+. Two functions $U(z) = U_1(z) - U_2(z)$ and $V(z) = V_1(z) - V_2(z)$, which are δ-subharmonic in a domain, are said to be equal, i.e., $U(z) = V(z)$, if $U_1(z) + V_2(z) = U_2(z) + V_1(z)$, everywhere in that domain.

We shall deal with the Tsuji characteristics of the form

$$\mathfrak{L}(y, \pm U) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (\pm U)^+(x + iy)dx + \int_y^{+\infty} n_\mp(t)dt, \quad 0 < y < +\infty,$$

where $a^+ = \max\{a, 0\}$, $a = a^+ - a^-$, and

$$n_\mp(t) = \iint_{G_t^+} d\nu_\mp(\zeta), \quad G_t^+ = \{\zeta : \operatorname{Im}\zeta > t\}.$$

Now, we introduce the ω-weighted classes of δ-subharmonic functions in G^+, which we shall study.

Definition 6.1. A δ-subharmonic in G^+ function $U(z)$ is of the class $\mathfrak{N}_\omega^m$, if

$$\sup_{y>\rho}[\mathfrak{L}(y, U) + \mathfrak{L}(y, -U)] < +\infty, \quad \text{for any} \quad \rho \in (0, \Delta), \tag{6.1}$$

and

$$\sup_{y>0}[\mathfrak{L}(y, L_\omega U) + \mathfrak{L}(y, -L_\omega U)] < +\infty. \tag{6.2}$$

Remark 6.1. In contrast to the theories in the unit disc of the complex plane [1, 2, 5] based on the equilibrium relation between the growth and the decrease Nevanlinna characteristics, such an equilibrium, i.e., Levin formula, is not true for all functions delta-subharmonic in the half-plane (see Chapter 3 in [6]). Therefore, it is natural to define the class $\mathfrak{N}_\omega^m$ by the restrictions (6.1) and (6.2), which are on both growth and decrease Tsuji characteristics, as it is done also in Chapter 4 of [6].

The following theorem gives the descriptive Riesz type representations of the classes $\mathfrak{N}_\omega^m$:

Theorem 6.1. (1°) *The class* $\mathfrak{N}_\omega^m$ *coincides with the set of functions of the form*

$$U(z) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \operatorname{Re}\{C_\omega(z-t)\}d\mu(t) + \iint_{G^+} \log|\widetilde{b}_\omega(z, \zeta)|d\nu(\zeta), \quad z \in G^+, \tag{6.3}$$

where $\mu(t)$ *is a function of bounded variation on* $(-\infty, +\infty)$ *and* $\nu(\zeta) = \nu_+(\zeta) - \nu_-(\zeta)$, *where* $\nu_\pm(\zeta)$ *are nonnegative Borel measures in* G^+, *such that*

$$\iint_{G^+}\left(\int_0^{\operatorname{Im}\zeta} \omega(x)dx\right)d\nu_\pm(\zeta) < +\infty. \tag{6.4}$$

(2°) *The measure* $\mu(t)$ *in representation* (6.3) *is revealed by the Stieltjes inversion formula*

$$\mu(x) = \lim_{y\to+0}\int_0^x L_\omega U_\omega(t+iy)dt \quad \text{a.e.} \quad x \in (-\infty, +\infty). \tag{6.5}$$

Proof. (1°) Let $U(z) \in \mathfrak{N}_\omega^m$. Then by (6.1) for any $\rho \in (0, \Delta)$,

$$\iint_{G_\rho^+}\left(\int_0^{\eta-\rho} \omega(x)dx\right)d\nu_\pm(\zeta) \le \iint_{G_\rho^+}\left(\int_0^{\eta} \omega(x)dx\right)d\nu_\pm(\zeta) < +\infty, \tag{6.6}$$

where $\zeta = \xi + i\eta$. Indeed, the first inequality is obvious. For proving the second one, observe that for any $\rho \in (0, \Delta)$,

$$\iint_{G_\rho^+}\left(\int_0^\eta \omega(x)dx\right)d\nu_\pm(\zeta)$$

$$=\left(\iint_{G_\Delta^+}+\iint_{G_\rho^+\setminus G_\Delta^+}\right)\left(\int_0^\eta \omega(x)dx\right)d\nu_\pm(\zeta)$$

$$\leq \iint_{G_\Delta^+}\left(\int_0^\Delta \omega(x)dx\right)d\nu_\pm(\zeta)+\omega(\Delta)\iint_{G_\rho^+\setminus G_\Delta^+}\eta d\nu_\pm(\zeta)$$

$$\leq \omega(\Delta)\iint_{G_\rho^+}\eta d\nu_\pm(\zeta)\leq \omega(\Delta)\iint_{G_{\rho/2}^+}\left(\eta-\frac{\rho}{2}\right)d\nu_\pm(\zeta)<+\infty,$$

since the condition (6.1) is true for $\rho/2$ in particular.

By (6.6) and Theorem 3.1, the Green type potentials in G_ρ^+, with the measures $\nu_\pm(\zeta)$, are convergent, and hence the function

$$U_0(z)=U(z)-\iint_{G_\rho^+}\log|\widetilde{b}_\omega(z-i\rho,\ \zeta-i\rho)|d\nu(\zeta),$$

is harmonic in G_ρ^+. Consequently, also the function

$$L_\omega U_0(z)=L_\omega U(z)-\iint_{G_\rho^+}L_\omega\log|\widetilde{b}_\omega(z-i\rho,\ \zeta-i\rho)|d\nu(\zeta), \tag{6.7}$$

is harmonic in G_ρ^+. Besides, by the continuity of $L_\omega U_0(z)$ and of the Green type potential, the function $L_\omega U(z)$ of the above formula is a continuous, δ-subharmonic function in G_ρ^+, and hence in the whole G^+. Further, it is obvious that

$$\sup_{y>0}\int_{-\infty}^{+\infty}|L_\omega U_0(x+iy)|dx$$

$$\leq \sup_{y>0}\int_{-\infty}^{+\infty}|L_\omega U(x+iy)|dx$$

$$+ \sup_{y>0} \int_{-\infty}^{+\infty} \left| \iint_{G_\rho^+} L_\omega \log|\widetilde{b}_\omega(x + iy - i\rho,\ \zeta - i\rho)|\, d\nu(\zeta) \right| dx$$

$$\equiv A + B,$$

where

$$A \le \sup_{y>\rho} [\mathfrak{L}(y,\ L_\omega U) + \mathfrak{L}(y,\ -L_\omega U)] < +\infty,$$

by (6.2) and $B < +\infty$ by the estimate (3.9). Thus,

$$L_\omega U_0\,(z + i\rho) = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{d\mu(t)}{(x - t)^2 + y^2}, \quad z = x + iy \in G^+,$$

where $\mu(t)$ is a function of bounded variation on $(-\infty, +\infty)$. Moreover, the measure $d\mu(t)$ of the above representation is absolutely continuous and is equal to $L_\omega U(t + i\rho)dt$. For proving this, observe that for any function $f(x)$ continuous in $(-\infty, +\infty)$ and such that $\lim_{x \to \pm\infty} f(x) = 0$

$$\lim_{y \to +0} \int_{-\infty}^{+\infty} f(x) L_\omega U_0 (x + iy + i\rho) dx = \int_{-\infty}^{+\infty} f(x) d\mu(x),$$

(see, e.g., [4], Chapter I, Theorems 5.3 and 3.1(c)). Then, for any interval $[a, b] \subset (-\infty, +\infty)$ introduce the sequence of functions $\{f_n(x)\}_1^\infty$ assuming that $f_n(x) \equiv 1\,(a \le x \le b)$ and $f_n(x) \equiv 0\,(x \notin [a - 1/n,\ b + 1/n])$ and continuing $f_n(x)$ to the remaining intervals of $(-\infty, +\infty)$ as a continuous, linear function. Then, by the continuity of $L_\omega U(t + i\rho)$ and the relation (3.10)

$$\int_{-\infty}^{+\infty} f_n(x) L_\omega U(x + i\rho) dx = \lim_{y \to +0} \int_{-\infty}^{+\infty} f_n(x) L_\omega U(x + iy + i\rho) dx$$

$$= \int_{-\infty}^{+\infty} f_n(x) d\mu(x),$$

for any $n = 1, 2, \ldots$. On the other hand,

$$\int_a^b L_\omega U(x + i\rho) dx = \lim_{n \to \infty} \int_{-\infty}^{+\infty} f_n(x) L_\omega U(x + i\rho) dx = \int_a^b d\mu(x).$$

Thus, $d\mu(t) \equiv L_\omega U(t+i\rho)dt$. Besides, $L_\omega U(t+i\rho) \in L^1(-\infty, +\infty)$ and

$$L_\omega U_0(z) = \frac{y-\rho}{\pi}\int_{-\infty}^{+\infty} \frac{L_\omega U(t+i\rho)}{(x-t)^2+y^2}\,dt, \quad z = x+iy \in G_\rho^+.$$

Consequently, by (6.7),

$$L_\omega U(z) - \iint_{G_\rho^+} L_\omega \log|\widetilde{b}_\omega(z-i\rho, \zeta-i\rho)|\,d\nu(\zeta) = \frac{y-\rho}{\pi}\int_{-\infty}^{+\infty} \frac{L_\omega U(t+i\rho)}{(x-t)^2+y^2}\,dt,$$

for any $z = x+iy \in G_\rho^+$, and

$$\frac{1}{2}\lim_{y\to+\infty} y\left\{L_\omega U(iy) - \iint_{G_\rho^+} L_\omega \log|\widetilde{b}_\omega(z-i\rho, \zeta-i\rho)|\,d\nu(\zeta)\right\}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} L_\omega U(t+i\rho)dt.$$

On the other hand, by the representation (2.10),

$$\frac{1}{2}\lim_{y\to+\infty} y\iint_{G_\rho^+} L_\omega \log|\widetilde{b}_\omega(z-i\rho, \zeta-i\rho)|\,d\nu(\zeta) = -\iint_{G_\rho^+}\left(\int_0^{\eta-\rho}\omega(t)dt\right)d\nu(\zeta),$$

where $\zeta = \xi + i\eta$. Thus,

$$\frac{1}{2}\lim_{y\to+\infty} yL_\omega U(iy) + \iint_{G_\rho^+}\left(\int_0^{\eta-\rho}\omega(t)dt\right)d\nu(\zeta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} L_\omega U(t+i\rho)dt, \tag{6.8}$$

where all quantities are finite.

Now, observe that by the relation (3.10) and a result of Solomentsev [10], the integrals in (6.7) taken by the components $d\nu_+(\zeta)$ and $d\nu_-(\zeta)$ of the measure $d\nu(\zeta)$ are ordinary Green potentials in G_ρ^+, i.e., there exist some nonnegative Borel measures $d\nu_\omega^{(+)}(\zeta)$ and $d\nu_\omega^{(-)}(\zeta)$ with non-overlapping supports in G_ρ^+, such that

$$\iint_{G_\rho^+} (\eta-\rho)d\nu_\omega^{(\pm)}(\zeta) < +\infty,$$

and

$$\iint_{G_\rho^+} L_\omega \log|\widetilde{b}_\omega(z - i\rho,\ \zeta - i\rho)|d\nu_\pm(\zeta) = \iint_{G_\rho^+} \log|b_0(z - i\rho,\ \zeta - i\rho)|d\nu_\omega^{(\pm)}(\zeta),$$

where $b_0(z, \zeta) = (z - \zeta)/(z - \overline{\zeta})$ is the ordinary Blaschke factor. Note that the measures $d\nu_\omega^{(\pm)}(\zeta)$ are independent of ρ, since they are the positive and the negative variations of the Riesz measure of the function $L_\omega U(z)$. Further, by a passage to the limit in the last formula, we obtain that for any $\rho \in (0, \Delta)$,

$$\iint_{G_\rho^+} \left(\int_0^{\eta - \rho} \omega(t)dt \right) d\nu_\pm(\zeta) = \iint_{G_\rho^+} (\eta - \rho) d\nu_\omega^{(\pm)}(\zeta) < +\infty. \tag{6.9}$$

Inserting this equality in (6.8), we come to the Levin formula for the function $L_\omega U(z)$. Then, by some simple rearrangement of terms, we get the following equilibrium relation for the Tsuji characteristics:

$$\frac{1}{2} \lim_{y \to +\infty} y L_\omega U(iy) + \mathfrak{L}(\rho, -L_\omega U) = \mathfrak{L}(\rho, L_\omega U), \quad 0 < \rho < \Delta.$$

Hence, by (6.2),

$$\iint_{G^+} \eta d\nu_\omega^{(\pm)}(\zeta) < +\infty,$$

which implies (6.4) in virtue of (6.9).

So, the relation (6.4) is true. Consequently, the Green type potential in (6.3) converges and the harmonic in G^+ function

$$U(z) - \iint_{G^+} \log|\widetilde{b}_\omega(z, \zeta)|d\nu(\zeta), \quad z \in G^+,$$

satisfies the conditions (5.1) and (5.2) of Theorem 5.1. Indeed, $U(z)$ satisfies (5.1) in virtue of (6.1), and the Green type potential by the estimate (4.7). As to the condition (5.2), $U(z)$ satisfies this condition by (5.2), and the Green type potential by the estimate (3.9). Thus, the considered function is of the form (5.3), and $U(z)$ is of the form (6.3).

Conversely, let $U(z)$ be representable in the form (6.3). Then, obviously, $U(z)$ is a δ-subharmonic function in G^+. Further, the Green type potential in (6.3) satisfies the condition (5.1) by (4.7) and it satisfies the condition (5.2) by (3.9). Besides, the function

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} \operatorname{Re}\{C_\omega(z-t)\}d\mu(t)$$

in (6.3), which is harmonic in G^+, satisfies these conditions by Theorem 5.1. Thus, $U(z) \in \mathfrak{N}_\omega^m$.

(2°) The relation (6.5) follows from (5.4) and (3.10), since

$$L_\omega U(z) = \frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{d\mu(t)}{(x-t)^2+y^2} + \iint_{G^+}\log|b_0(z,\zeta)|d\nu_\omega(\zeta), \quad z = x+iy \in G^+,$$

where $\nu_\omega(\zeta)$ is a Borel measure such that its positive and negative variations satisfy the condition

$$\iint_{G^+}\operatorname{Im}\zeta d\nu_\omega^{(\pm)}(\zeta) < +\infty.$$

Remark 6.2. In particular, the above theorem implies that the class of those functions $f(z)$ meromorphic in G^+, for which $\log|f(z)| \in \mathfrak{N}_\omega^m$, coincides with the set of functions representable in the form

$$f(z) = \frac{\widetilde{B}(z,\{a_k\})}{\widetilde{B}(z,\{b_n\})}\exp\left\{\frac{1}{\pi}\int_{-\infty}^{+\infty}C_\omega(z-t)d\mu(t)+iC\right\}, \quad z \in G^+, \tag{6.10}$$

where $\mu(t)$ is a function of bounded variation on $(-\infty, +\infty)$ C is a real number and $\{a_k\} \subset G^+$, $\{b_n\} \subset G^+$ are the zeros and the poles of $f(z)$, which satisfy the density condition

$$\sum_k\int_0^{\operatorname{Im} a_k}\omega(t)dt < +\infty, \quad \sum_n\int_0^{\operatorname{Im} b_n}\omega(t)dt < +\infty.$$

If the above factorization of $f(z)$ is true, the following analogue of the Stieltjes inversion formula is valid:

$$\mu(x) = \lim_{y \to +0} \int_0^x L_\omega \log|f(t + iy)|\,dt, \quad \text{a.e.} \quad x \in (-\infty, +\infty).$$

Remark 6.3. Note that a change of the integration orders in the exponent of the factorization (6.10) and in the harmonic part of the representation (6.3) gives a Laplace transform and the real part of a Laplace transform.

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